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Spin-up of a strongly stratified fluid in a sphere

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A linear theory is developed for the spin-up of a compressible fluid, stratified by a spherical gravity field. Numerical results are obtained for the case of strong stratification (Brunt–Väisälä frequency N much greater than the rotation frequency Ω_0). The interior flow is solved in terms of a set of angular eigenfunctions which have been obtained numerically. The principal result is that the spin-up is limited to a layer adjacent to the spherical boundary, the thickness δ of the layer being of the order of $L(\Omega_0/N)$, where L is the radius of the boundary. The solution is qualitatively similar to that found by Holton (1965), Walin (1969), and Sakurai (1969*a*, *b*) for a stratified fluid in a cylinder. The thickness of the spin-up layer diminishes with latitude ϕ, the variation being described roughly by the formula $\delta \sim L\Omega_0|\sin\phi|/N$. For the case of slow continuous spin-up, the Ekman suction velocity has been calculated, and the results show that $|\phi| = 24°$ is the dividing angle between suction ($|\phi| > 24°$) and blowing ($|\phi| < 24°$).

1. Introduction

An important problem in the theory of rotating fluids is the so-called spin-up problem, in which one analyzes the motions induced in an enclosed, uniformly rotating fluid by a sudden increase in angular velocity of the container. The first complete analysis of such a problem was given by Greenspan & Howard (1963) for the case of an incompressible, unstratified fluid. For a stratified fluid, there are two general types of spin-up problems: (i) laboratory flows, in which the gravity field is constant, and (ii) geophysical flows in which the gravity equipotentials are closed surfaces (typically spherical or nearly so). A prototype of laboratory flows is the case of the circular cylinder which has been studied by Holton (1965), Sakurai (1969*a*, *b*) and Walin (1969). In the present work, we analyze the simplest of the geophysical flows, namely the spin-up of a stratified fluid in a sphere.

The analysis given here rests on four major assumptions. The first is linearization about a state of uniform rotation. (As is well known, there are necessarily small, thermally driven circulations in a rotating, stratified gas. We assume that the rotation is slow enough for these circulations to be negligible, and we do not consider them further in the present work.) The second is the assumption of geostrophic flow, or, alternatively, the assumption that inertial modes are not excited. This means that the time-scale for variations of the boundary angular velocity should be either much shorter than or much longer than the basic rotation period. In particular, this includes the important case of impulsive spin-up. Our third assumption is that the stratification is strong. Specifically, we take $(\Omega_0/N) \ll 1$, where Ω_0 is the basic rotation frequency and N is the Brunt–Väisälä frequency. Further, we assume that (Ω_0/N) is much less than the ratio of the scale height to a typical dimension of the container. As we shall see later, these assumptions allow a great simplification of the mathematical problem. The fourth assumption is that the container boundary is an equipotential surface for the total (gravity plus centrifugal) force field. This assumption, which seems reasonable for problems of geophysical interest, has the consequence that the thermal and viscous boundary layers are uncoupled. It appears possible to analyze the spin-up problem without this assumption, but some features of interest would be obscured by technical complexities. This point is discussed further in §2.3.

The formulation of the problem, on the basis of the above assumptions, is carried out in §2. In §2.2 we derive the basic second-order elliptic equation which governs the interior flow. The boundary condition on the interior flow is obtained from the boundary-layer analysis of §2.3. The general analysis of §2 (valid for an arbitrary axisymmetric container) is specialized to the case of the sphere in §3. We show in §3.1 that the approximations based on strong stratification allow a separation-of-variables solution of the governing equation. The angular eigenfunctions arising in the separation are not standard special functions. They have been obtained numerically, however, and appendix A of this paper describes their computation and use. (In addition, appendix A contains an analytical asymptotic analysis which provides a check on the numerical theory.) An evaluation and discussion of the solution is given in §3.2.

The present work is part of a program of calculations which are intended to elucidate the solar spin-down problem. This program, along with our point-of-view on the solar problem is described elsewhere (Clark, Thomas & Clark 1969). A brief discussion of the implications of the present work for the solar problem is given in §4 below.

2. Formulation

In addition to the assumptions listed in §1, there are a number of minor approximations which have been made to reduce the mathematical complexity of the problem. In order to make clearer the detailed discussion of the equations below, we give a summary here of the general features of the flow field. The justification for the claims made in this summary rests mostly on what is already known about rotating fluids (Greenspan 1968).

Let ν be a characteristic kinematic viscosity of the fluid, L a characteristic dimension of the container, and Ω_0 the angular velocity of the uniform rotation. Then the basic small parameter in the problem is the Ekman number,

$$E = \nu/(L^2\Omega_0). \tag{1}$$

The characteristic time for changes in the interior geostrophic flow is $\Omega_0^{-1}E^{-\frac{1}{2}}$. On this time scale, the direct influence of viscous diffusion is confined to a layer adjacent to the boundary with thickness of the order $LE^{\frac{1}{2}}$ (Ekman layer), and the thermal effects are confined to a layer with thickness of the order of $LE^{\frac{1}{4}}$. (We assume throughout this work that the Prandtl number of the fluid is of order one with respect to the Ekman number.) A second important parameter is the ratio Ω_0/N, where N is a characteristic value of the Brunt–Väisälä frequency. We assume that Ω_0/N is small, but that it is of the order of one with respect to $E^{\frac{1}{2}}$. The results of our calculations show that the fluid appreciably affected by changes in the boundary angular velocity resides in a layer adjacent to the boundary with a thickness.

$$\delta \sim (\Omega_0/N)L. \tag{2}$$

This layer is much thicker than the viscous and thermal layers, but still thin compared to the container dimension.

2.1. *Basic equations*

Because of the large number of variables, we adopt the following systematic notation: (i) a subscript zero refers to the basic undisturbed state of uniform rotation; (ii) a superior caret means a dimensional variable; (iii) quantities without carets are dimensionless.

The undisturbed state is specified by the angular velocity $\hat{\mathbf{\Omega}}_0$, the pressure $\hat{p}_0$, the density $\hat{\rho}_0$, and the gravitational potential $\hat{\Phi}_0$, where $\hat{\Phi}_0$ is taken to be the sum of the centrifugal potential $-\frac{1}{2}|\hat{\mathbf{\Omega}}_0 \times \hat{\mathbf{R}}|^2$ and the true gravitational potential. The basic equation of hydrostatic balance is

$$\hat{\nabla}\hat{p}_0 = -\hat{\rho}_0\hat{\nabla}\hat{\Phi}_0. \tag{3}$$

As is well known, it follows from (3) that the level surfaces of $\hat{p}_0$, $\hat{\rho}_0$ and $\hat{\Phi}_0$ coincide. It then follows that the remaining thermodynamic variables, such as the temperature $\hat{T}_0$ and entropy per unit mass $\hat{s}_0$, are also constant on the equipotentials. (In making this claim, we assume that the functional form of the equation of state does not depend on position. The primary case of physical interest excluded by this assumption is the case of variable mean molecular weight.)

Consider now a disturbance described by the velocity $\hat{\mathbf{v}}$, the pressure $\hat{p}_0+\hat{p}$, the density $\hat{\rho}_0+\hat{\rho}$, and the entropy $\hat{s}_0+\hat{s}$. The linearized equations of mass, momentum, and energy (referred to the rotating frame) are

$$\frac{\partial\hat{\rho}}{\partial\hat{t}} + \hat{\nabla}\,.\,(\hat{\rho}_0\hat{\mathbf{v}}) = 0, \tag{4}$$

$$\hat{\rho}_0\frac{\partial\hat{\mathbf{v}}}{\partial\hat{t}} = -\hat{\nabla}\hat{p} - \hat{\rho}\hat{\nabla}\hat{\Phi}_0 - 2\hat{\rho}_0\hat{\mathbf{\Omega}}_0\times\hat{\mathbf{v}} + \hat{\rho}_0\hat{\mathbf{f}}, \tag{5}$$

and

$$\hat{\rho}_0\hat{T}_0\left(\frac{\partial\hat{s}}{\partial\hat{t}} + \hat{\mathbf{v}}\,.\,\hat{\nabla}\hat{s}_0\right) = -\hat{\nabla}\,.\,\hat{\mathbf{Q}}. \tag{6}$$

Here $\hat{\mathbf{f}}$ is the viscous force per unit mass, and $\hat{\mathbf{Q}}$ is the perturbation to the heat flux vector. We have neglected the perturbation to the gravity associated with the density perturbation. The justification is based on the fact that the radial extent

$\hat{\delta}$ of the perturbation is much smaller than the extent $\hat{L}$ of the body—$\hat{\delta}/\hat{L} \sim \hat{\Omega}_0/\hat{N} \ll 1$. It can then be shown that the contribution of the perturbed gravity field to the total force is smaller than $\hat{\rho}\hat{\nabla}\hat{\Phi}_0$ by the ratio $\hat{\Omega}_0/\hat{N}$. To close the set of equations (4)–(6), we need a thermodynamic relation connecting $\hat{p}$, $\hat{\rho}$, and $\hat{s}$. The general equation, correct to first order, is

$$\hat{p} = \hat{c}_0^2(\hat{\rho} + \hat{\rho}_0\hat{\beta}_0\hat{T}_0\hat{s}/\hat{c}_{p_0}), \tag{7}$$

where

$$\hat{c}_0^2 = (\partial\hat{p}_0/\partial\hat{\rho}_0)_s \tag{8}$$

is the square of the sound speed, $\hat{c}_{p_0}$ is the specific heat at constant pressure, and

$$\hat{\beta}_0 = -\hat{\rho}_0^{-1}(\partial\hat{\rho}_0/\partial\hat{T}_0)_p \tag{9}$$

is the coefficient of thermal expansion.

Equations (4)–(6) apply to a wide class of motions, and they can be simplified considerably in the present special problem. Consider first the transport terms $\hat{\rho}_0\hat{\mathbf{f}}$ and $\hat{\nabla}\cdot\hat{\mathbf{Q}}$, which are, in general, very complicated for a compressible fluid with variable transport properties. During the time scale of interest, the diffusive effects are confined to thin boundary layers near the wall. The variation of the transport coefficients across these layers will be slight, and we may approximate the terms by

$$\hat{\mathbf{f}} = \hat{\nu}\hat{\nabla}^2\hat{\mathbf{v}} \tag{10}$$

and

$$-\hat{\nabla}\cdot\hat{\mathbf{Q}} = \hat{\rho}_0\hat{c}_{p_0}\hat{\chi}\hat{\nabla}^2\hat{T}, \tag{11}$$

where the viscosity $\hat{\nu}$ and thermal diffusivity $\hat{\chi}$ are now taken to be constant. The difference between these approximations and the actual expressions is small in the boundary layers and unimportant elsewhere.

Further simplifications are possible. The fact that we are dealing with small-amplitude, slow motions suggests that the perturbations may be very nearly incompressible. This turns out to be so, but the justifying order-of-magnitude estimates are rather lengthy, so they are presented in appendix B. The result is that the continuity equation (4) may be replaced by the much simpler equation $\hat{\nabla}\cdot\hat{\mathbf{v}} = 0$. This in turn may be exploited by the introduction of a stream function $\hat{\psi}$, so that the axisymmetric velocity field may be written as

$$\hat{\mathbf{v}} = \hat{\nabla}\times(\mathbf{e}_\phi\hat{\psi}/\hat{r}) + \hat{w}\mathbf{e}_\phi, \tag{12}$$

where $\mathbf{e}_\phi$ is a unit vector in the azimuthal direction, $\hat{r}$ is the cylindrical radial co-ordinate, and $\hat{w}$ is the azimuthal velocity. Since the flow is axisymmetric, both $\hat{w}$ and $\hat{\psi}$ are independent of the azimuthal angle ϕ. With these simplifications, the basic equations take the following form:

$$\hat{\nabla}\left[\frac{\partial\hat{\psi}}{\partial\hat{t}} - \hat{\nu}\left\{\hat{r}\hat{\nabla}^2\left(\frac{\hat{\psi}}{\hat{r}}\right) - \frac{\hat{\psi}}{\hat{r}^2}\right\}\right]\times\frac{\mathbf{e}_\phi}{\hat{r}} = -2\mathbf{\Omega}_0\times\mathbf{e}_\phi\hat{w} - \frac{\hat{\nabla}\hat{p}}{\hat{\rho}_0} - \frac{\hat{\rho}}{\hat{\rho}_0}\hat{\nabla}\hat{\Phi}_0, \tag{13}$$

$$\frac{\partial\hat{w}}{\partial\hat{t}} = \frac{2}{\hat{r}}\hat{\mathbf{\Omega}}_0\cdot\hat{\nabla}\hat{\psi} + \hat{\nu}\left(\hat{\nabla}^2\hat{w} - \frac{\hat{w}}{\hat{r}^2}\right), \tag{14}$$

and

$$\frac{\partial\hat{s}}{\partial\hat{t}} + \frac{\mathbf{e}_\phi}{\hat{r}}\cdot\hat{\nabla}\hat{s}_0\times\hat{\nabla}\hat{\psi} = \frac{\hat{c}_{p_0}\hat{\chi}}{\hat{T}_0}\hat{\nabla}^2\hat{T}. \tag{15}$$

It will be convenient later to express $\hat{\nabla}\hat{s}_0$ in terms of the Brunt–Väisälä frequency $\hat{N}$, where

$$\hat{N}^2 = (\hat{\beta}_0\hat{T}_0/\hat{c}_{p_0})\hat{\nabla}\hat{\Phi}_0\cdot\hat{\nabla}\hat{s}_0. \tag{16}$$

(The physical interpretation of $\hat{N}$ is the oscillation frequency of a small element of fluid displaced adiabatically from equilibrium.) Since $\hat{s}_0$ is a function of $\hat{\Phi}_0$ alone, one can show from (16) that

$$\hat{\nabla}\hat{s}_0 = \frac{\hat{c}_{p_0}\hat{N}^2}{\hat{\beta}_0\hat{T}_0|\hat{\nabla}\hat{\Phi}_0|^2}\hat{\nabla}\hat{\Phi}_0. \tag{17}$$

The first step in the analysis of these equations is the proper scaling of quantities. The scaling must be correct with respect to the Ekman number, since it is the basic small parameter in the expansion scheme. The scaling with respect to the other parameters is largely a matter of convenience, since they are all of order one with respect to the Ekman number. Lengths are scaled by $\hat{L}$, the characteristic container dimension, and for the time scale, we make use of the known result that spin-up occurs in a time of the order of $\hat{\Omega}_0^{-1}E^{-\frac{1}{2}}$. The dimensionless independent variables and gradient operator are then

$$t = E^{\frac{1}{2}}\hat{\Omega}_0\hat{t}, \quad \mathbf{R} = \hat{L}^{-1}\hat{\mathbf{R}}, \quad \nabla = \hat{L}\hat{\nabla}.$$

For the azimuthal velocity $\hat{w}$, we choose the scale $\hat{L}\hat{\Omega}_0$. Then we use the known result that the interior circulations are $O(E^{\frac{1}{2}})$ with respect to $\hat{w}$ to get the scaling for $\hat{\psi}$. The dimensionless azimuthal velocity and stream function are

$$w = (\hat{L}\hat{\Omega}_0)^{-1}\hat{w}, \quad \psi = (E^{\frac{1}{2}}\hat{L}^3\hat{\Omega}_0)^{-1}\hat{\psi}.$$

The proper scaling for the pressure is established by considering the force balance along an equipotential. On this basis, we define the dimensionless pressure,

$$p = (\hat{L}^2\hat{\Omega}_0^2\hat{\rho}_*)^{-1}\hat{p},$$

where $\hat{\rho}_*$ is a constant characteristic value of $\hat{\rho}_0$. For the other thermodynamic quantities, we use the following scaling:

$$\rho = \hat{\rho}_*^{-1}\hat{\rho}, \quad T = \hat{T}_0^{-1}\hat{T}, \quad s = \hat{c}_{p_0}^{-1}\hat{s}.$$

Finally, the gravitational potential must be scaled properly. We let $\hat{g}_*$ be a (constant) characteristic value of $|\hat{\nabla}\hat{\Phi}_0|$. Then the dimensionless potential is

$$\Phi_0 = (\hat{L}\hat{g}_*)^{-1}\hat{\Phi}_0.$$

The dimensionless gravity is $g = \hat{g}/\hat{g}_*$, where $g = |\nabla\Phi_0|$ and $\hat{g} = |\hat{\nabla}\hat{\Phi}_0|$. With the above scalings, the basic equations (13)–(15) can be put into the form,

$$E\nabla\left[\frac{\partial\psi}{\partial t} - E^{\frac{1}{2}}\left\{r\nabla^2\left(\frac{\psi}{r}\right) - \frac{\psi}{r^2}\right\}\right]\times\frac{\mathbf{e}_\phi}{r} = 2w\mathbf{e}_r - \frac{\nabla p}{\rho_0} - G\frac{\rho}{\rho_0}\nabla\Phi_0, \tag{18}$$

$$\frac{\partial w}{\partial t} = \frac{2}{r}\mathbf{e}_z.\nabla\psi + E^{\frac{1}{2}}\left(\nabla^2 w - \frac{w}{r^2}\right), \tag{19}$$

and

$$\frac{\partial s}{\partial t} + \frac{1}{r}(\nabla H_0 \times \nabla\psi).\mathbf{e}_\phi = \mathscr{P}^{-1}E^{\frac{1}{2}}\nabla^2 T, \tag{20}$$

where $\mathbf{e}_r$ and $\mathbf{e}_z$ are unit vectors in the r- and z-directions, and where $\rho_0 = \hat{\rho}_0/\hat{\rho}_*$,

$\mathscr{P} = \hat{\nu}/\hat{\chi}$ is the Prandtl number, and $G = \hat{g}_*/(\hat{L}\hat{\Omega}_0^2)$ is the inverse of the Froude number. The quantity H_0 is a dimensionless function of the basic stratification, defined by

$$H_0(\Phi_0) = \int \frac{1}{\hat{c}_{p_0}} \frac{d\hat{s}_0}{d\hat{\Phi}_0}\, d\hat{\Phi}_0. \tag{21}$$

A formula needed later is

$$\frac{dH_0}{d\Phi_0} = \frac{\hat{N}^2\hat{L}}{\hat{\beta}_0 \hat{T}_0 g^2 \hat{g}_*}, \tag{22}$$

which is obtained from (17) and (21). In (20) we have replaced $\hat{T}_0^{-1}\nabla^2(\hat{T}_0 T)$ by $\nabla^2 T$, an approximation which is justified by the thinness of the diffusive boundary layers. The dimensionless form of the thermodynamic relation (7) is

$$p = (\hat{c}_0/\hat{L}\hat{\Omega}_0)^2(\rho + \rho_0\hat{\beta}_0\hat{T}_0 s). \tag{23}$$

2.2. *Interior equations*

By setting $E = 0$ in (18)–(20), we obtain the equations for the interior flow,

$$2\rho_0 w \mathbf{e}_r = \nabla p + G\rho\nabla\Phi_0, \tag{24}$$

$$(\partial w/\partial t) = (2/r)\mathbf{e}_z . \nabla\psi, \tag{25}$$

and

$$(\partial s/\partial t) + r^{-1}\mathbf{e}_\phi . (\nabla H_0 \times \nabla\psi) = 0. \tag{26}$$

It is advantageous to combine these into a single equation for ψ. The first step is to use (23) to eliminate ρ from (24). After a little calculation, the resulting equation may be put into the form

$$\rho_0 F(2w\nabla r + G\hat{\beta}_0\hat{T}_0 s\nabla\Phi_0) = \nabla(Fp), \tag{27}$$

where we have replaced $\mathbf{e}_r$ by ∇r, and where F is a function of the basic stratification given by

$$F(\Phi_0) = \exp\left[\int \frac{d\hat{\Phi}_0}{\hat{c}_0^2}\right].$$

The pressure p may be eliminated by taking the curl of (27) to get

$$\nabla r \times \nabla(2\rho_0 F w) + G\hat{\beta}_0\hat{T}_0\rho_0 F\nabla\Phi_0 \times \nabla s = 0, \tag{28}$$

where we have used the fact that $\hat{\beta}_0\hat{T}_0\rho_0 F$ is a function of Φ_0 alone. Equation (28) is a time-independent constraint which, roughly speaking, relates the variation of w along cylindrical surfaces r = constant to the variation of s along equipotential surfaces Φ_0 = constant. The variations in w (equation (25)) are determined by the cylindrical radial velocity, whereas the variations in s (equation (26)) are determined by the velocity normal to equipotential surfaces. Thus, the ratio of the changes in w and s will depend on the flow direction, and this degree of freedom means that in general the constraint (28) can be satisfied. (The flow near any internal surface where $\mathbf{\Omega}_0 . \nabla\Phi_0 = 0$ is an important exception. In that case, the relevant directions—perpendicular to $\mathbf{\Omega}_0$, parallel to $\nabla\Phi_0$—coincide. As we shall see later, this has the consequence that the spin-up in the equatorial plane is much smaller than elsewhere.)

Before completing the reduction to a single equation, we introduce some simplifying notation: for any scalars A, B, let

$$\mathbf{e}_\phi . \nabla A \times \nabla B = \langle A, B \rangle.$$

Then (25), (26), and (28) may be written as

$$(\partial w/\partial t) = (2/r)\langle \psi, r \rangle, \tag{29}$$

$$(\partial s/\partial t) = r^{-1}\langle \psi, H_0 \rangle, \tag{30}$$

and

$$\langle r, 2\rho_0 F w \rangle + G\hat{\beta}_0 \hat{T}_0 \rho_0 F \langle \Phi_0, s \rangle = 0. \tag{31}$$

A single equation for ψ is obtained by taking the time-derivative of (31) and then using (29) and (30) to get

$$\left\langle r, \frac{4\rho_0 F}{r}\langle \psi, r \rangle \right\rangle + G\hat{\beta}_0 \hat{T}_0 \rho_0 F \left\langle \Phi_0, \frac{1}{r}\langle \psi, H_0 \rangle \right\rangle = 0.$$

We may simplify this somewhat by using (22) for $dH_0/d\Phi_0$ and the identities

$$\langle A, BC \rangle = \langle A, B \rangle C + B\langle A, C \rangle,$$

$$\langle A, F(B) \rangle = \langle A, B \rangle F'(B),$$

and

$$\langle A, F(A) \rangle = 0.$$

The result is

$$\left(\frac{2\hat{\Omega}_0 g}{\hat{N}}\right)^2 \left\langle r, \frac{\rho_0 F}{r}\langle \psi, r \rangle \right\rangle + \left\langle \Phi_0, \frac{\rho_0 F}{r}\langle \psi, \Phi_0 \rangle \right\rangle = 0. \tag{32}$$

This second-order elliptic equation governs the stream function for the interior flow. In order to solve the equation, it is necessary to specify the gravitational potential Φ_0, the 'structure functions' $\hat{N}$ and F, and a boundary condition on ψ. The boundary condition can be obtained through an analysis of the diffusive boundary layers, which we consider next.

2.3. *Boundary-layer analysis*

In general the interior solution will not satisfy the thermal and viscous boundary conditions at the container wall, and there will be thin diffusive boundary layers. The nature of the boundary layers depends strongly on the angle between the gravity vector and the wall normal. Hsueh (1969) has shown that whenever this angle greatly exceeds $E^{\frac{1}{2}}$, the viscous and thermal layers are coupled into a single layer of thickness $E^{\frac{1}{3}}$. If, however, gravity is parallel to the normal (i.e. angle $\ll E^{\frac{1}{2}}$), then one gets the usual $E^{\frac{1}{4}}$ thermal layer and $E^{\frac{1}{2}}$ Ekman layer. We have assumed in the present work that the gravity vector and the wall normal are parallel. As we shall see below, this allows a separation of the thermal and viscous boundary layers, with the result that the interior flow is independent of the thermal boundary condition.

In the investigation of the boundary layer, we use the following co-ordinate system: $\mathbf{e}_\alpha$ is a unit vector normal to the wall (pointing into the fluid), and α is distance measured along $\mathbf{e}_\alpha$; $\mathbf{e}_\beta$ is a unit vector tangent to the wall, and β is distance along the wall from some reference point. Because of the double layer structure ($E^{\frac{1}{2}}$ and $E^{\frac{1}{4}}$) we need two scaled normal co-ordinates,

$$\eta = \alpha/E^{\frac{1}{2}}, \quad \xi = \alpha/E^{\frac{1}{4}}.$$

Consider now the expansion procedure. Any physical quantity Λ will have a representation of the form

$$\Lambda = \Lambda^{(I)}(\alpha,\beta,E)+\Lambda^{(T)}(\xi,\beta,E)+\Lambda^{(E)}(\eta,\beta,E),$$

where $\Lambda^{(I)}$ is the interior solution, $\Lambda^{(T)}$ is the thermal layer correction, and $\Lambda^{(E)}$ is the Ekman layer correction. The corrections are assumed to be transcendentally small outside of their respective boundary layers. Each of the three functions in the representation has an expansion in powers of $E^{\frac{1}{4}}$ (at least for the first few terms); thus,

$$\Lambda^{(I)} = \Lambda_0^{(I)}(\alpha,\beta)+E^{\frac{1}{4}}\Lambda_1^{(I)}(\alpha,\beta)+E^{\frac{1}{2}}\Lambda_2^{(I)}(\alpha,\beta)+\ldots,$$

$$\Lambda^{(T)} = \Lambda_0^{(T)}(\xi,\beta)+E^{\frac{1}{4}}\Lambda_1^{(T)}(\xi,\beta)+E^{\frac{1}{2}}\Lambda_2^{(T)}(\xi,\beta)+\ldots,$$

and

$$\Lambda^{(E)} = \Lambda_0^{(E)}(\eta,\beta)+E^{\frac{1}{4}}\Lambda_1^{(E)}(\eta,\beta)+E^{\frac{1}{2}}\Lambda_2^{(E)}(\eta,\beta)+\ldots.$$

The procedure is to substitute expansions of this form into the basic equations (18)–(20), and then equate the coefficients of like powers of $E^{\frac{1}{4}}$. We consider only the equations for the zero-order quantities $\Lambda_0^{(I)}$, $\Lambda_0^{(T)}$, $\Lambda_0^{(E)}$, and, for simplicity, we omit the subscript zero in what follows. One gets three sets of equations, corresponding to the regions $\alpha = O(1)$, $\xi = O(1)$, and $\eta = O(1)$. The interior equations ($\alpha = O(1)$) are simply (24)–(26), as one would expect. The thermal layer equations ($\xi = O(1)$) are

$$\mathbf{e}_z\,.\,\mathbf{e}_\alpha(\partial\psi^{(T)}/\partial\xi) = 0 \quad (\text{thus, } \psi^{(T)} = 0),$$

$$(\partial p^{(T)}/\partial\xi) = 0 \quad (\text{thus, } p^{(T)} = 0),$$

$$2w^{(T)}\,\mathbf{e}_\beta\,.\,\nabla r = 0 \quad (\text{thus, } w^{(T)} = 0),$$

and

$$(\partial s^{(T)}/\partial t) = \mathscr{P}^{-1}(\partial^2 T^{(T)}/\partial\xi^2). \tag{33}$$

The Ekman layer equations ($\eta = O(1)$) are

$$(2/r)\,(\partial\psi^{(E)}/\partial\eta)\,\mathbf{e}_z\,.\,\mathbf{e}_\alpha = -\,(\partial^2 w^{(E)}/\partial\eta^2), \tag{34}$$

$$2w^{(E)}\,\mathbf{e}_\alpha\,.\,\mathbf{e}_z = r^{-1}(\partial^3\psi^{(E)}/\partial\eta^3), \tag{35}$$

$$(\partial p^{(E)}/\partial\eta) = 0 \quad (\text{thus, } p^{(E)} = 0),$$

and

$$(\partial^2 T^{(E)}/\partial\eta^2) = 0 \quad (\text{thus, } T^{(E)} = 0).$$

The thermal layer equations show that ψ, p, and w do not change across the $E^{\frac{1}{4}}$ layer. Since $p^{(T)} = 0$, the temperature and entropy corrections are related by

$$\hat{s}^{(T)} = (\partial\hat{s}_0/\partial\hat{T}_0)_p\,\hat{T}^{(T)} = (\hat{c}_{p_0}/\hat{T}_0)\,\hat{T}^{(T)},$$

so that $s^{(T)} = T^{(T)}$. Thus, (33) becomes a diffusion equation for $T^{(T)}$. The solution to this equation may be used to satisfy whatever thermal boundary condition is imposed at the wall. Since this thermal boundary layer has no effect on the motion to this order, we do not consider it further.

The Ekman layer equations show that the thermodynamic quantities do not vary across the $E^{\frac{1}{2}}$-layer. The two equations for $\psi^{(E)}$ and $w^{(E)}$ may be written as

$$2\cos\theta(\partial\psi^{(E)}/\partial\eta) = r(\partial^2 w^{(E)}/\partial\eta^2) \tag{36}$$

and

$$(\partial^3\psi^{(E)}/\partial\eta^3) = -2r\cos\theta w^{(E)}, \tag{37}$$

where θ is the angle from the vertical $\mathbf{e}_z$ to the exterior wall normal $-\mathbf{e}_\alpha$. The boundary conditions are that ψ and its normal derivative should vanish at the wall, and the azimuthal velocity should match the prescribed boundary velocity $r\Omega_B(t)$. Thus, we have (correct to the lowest order in the $E^{\frac{1}{4}}$ expansion)

$$\psi^{(E)}(\eta = 0, \beta) = -\psi^{(I)}(\alpha = 0, \beta), \tag{38}$$

$$(\partial\psi^{(E)}/\partial\eta) = 0, \tag{39}$$

and

$$w^{(E)}(\eta = 0, \beta) + w^{(I)}(\alpha = 0, \beta) = r\Omega_B. \tag{40}$$

The solution of (36) and (37), which satisfies the boundary conditions (38) and (39), is the familiar Ekman spiral:

$$\psi^{(E)} = -\psi^{(I)} e^{-d\eta}\{\cos(d\eta) + \sin(d\eta)\}$$

and

$$w^{(E)} = (2\sigma d/r)\psi^{(I)} e^{-d\eta}\cos(d\eta),$$

where $d = |\cos\theta|^{\frac{1}{2}}$ and $\sigma =$ sign of $\cos\theta$. The boundary condition (40) then yields the fundamental relation between the interior quantities:

$$2d\sigma\psi^{(I)} + rw^{(I)} = r^2\Omega_B. \tag{41}$$

The solution for the interior flow may be obtained from the equations of §2.2 and the boundary condition (41).

Perhaps it should be noted that the Ekman layer analysis given above breaks down very near the equator. Although the proper scaling for the equatorial Ekman layer has been given by Carrier (1965) and Stewartson (1966), the solution of the difficult mathematical problem is not yet available. In any case, the breakdown does not seem to have any serious consequences for the present problem.

3. Spherical container

3.1. *Calculation of the interior flow*

We now specialize to the case of a spherical container with a spherically symmetric stratification and gravity field. (We assume that the flattening due to the basic rotation is small enough to be ignored.) The solution of the equations is based on the smallness of the parameter $(2\hat{\Omega}_0/\hat{N})$, and only the first approximation is calculated here. A brief discussion of a rational expansion scheme in this parameter is given in §3.2.

The quantities of interest are the stream function ψ and azimuthal velocity w. The relevant equations are (29) and (32), with the boundary condition (41). The first step in the solution is the introduction of spherical co-ordinates R (radius) and θ (polar angle). We have

$$\langle A,B\rangle = \mathbf{e}_\phi.\nabla A\times\nabla B = \frac{1}{R}\left(\frac{\partial A}{\partial R}\frac{\partial B}{\partial\theta}-\frac{\partial A}{\partial\theta}\frac{\partial B}{\partial R}\right),$$

and (29) and (32) may be written as

$$(\partial w/\partial t) = (2/R\sin\theta)L\psi \tag{42}$$

and

$$\left(\frac{2\hat{\Omega}_0}{\hat{N}}\right)^2 L(\rho_0 FL\psi)+\frac{\rho_0 F}{R^2}\sin\theta\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial\psi}{\partial\theta}\right) = 0, \tag{43}$$

where

$$L = \cos\theta(\partial/\partial R)-(\sin\theta/R)(\partial/\partial\theta).$$

In (43) ρ_0, F, and $\hat{N}$ are all functions of R only. The quantity $(\hat{N}/2\hat{\Omega}_0)$ is large, and it follows from an order-of-magnitude analysis of (43) that the length scale for variations normal to the boundary is of the order of $(2\hat{\Omega}_0/\hat{N})$. It is one of our basic assumptions that this quantity is smaller than the scale height. Thus, the R-dependent coefficients in (43) do not vary much over the radial extent of the flow, and we can replace them with their values at the boundary $R = 1$. For the same reasons, $(\partial/\partial R) \gg (1/R)$, and we can replace the operator L by $\cos\theta(\partial/\partial R)$. It is convenient to introduce the large parameter,

$$B = (\hat{N}/2\hat{\Omega}_0)|_{R=1},$$

and the new normal co-ordinate,

$$\zeta = B(1-R).$$

Then, the approximate equations for w and ψ may be written as

$$(\partial w/\partial t) = -2B\cot\theta(\partial\psi/\partial\zeta) \tag{44}$$

and

$$\cos^2\theta\frac{\partial^2\psi}{\partial\zeta^2}+\sin\theta\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial\psi}{\partial\theta}\right) = 0. \tag{45}$$

We now solve (44) and (45) subject to the boundary condition (41). The solution is readily obtained by separation of variables, and is conveniently written in terms of $x = \cos\theta$:

$$w = x\sum_{n=1}^{\infty} c_n(t)\exp(-\lambda_n\zeta)\,w_n(x),$$

and

$$\psi = \frac{(1-x^2)^{\frac{1}{2}}}{2B}\sum_{n=1}^{\infty}\frac{1}{\lambda_n}\frac{dc_n}{dt}\exp(-\lambda_n\zeta)\,w_n(x).$$

Here w_n and λ_n are the eigenfunctions and eigenvalues of the Sturm–Liouville system,

$$\frac{d}{dx}\left[(1-x^2)\frac{dw_n}{dx}\right]-\frac{w_n}{1-x^2} = -\lambda_n^2 x^2 w_n, \tag{46}$$

with $w_n(0) = 0$ (ψ is an odd function of x) and w_n well-behaved at the pole $x = 1$. The numerical solution of this eigenvalue problem is described in appendix A, along with some of the basic properties of the eigenfunctions. The amplitude of w_n has been fixed by the normalization

$$\int_0^1 x^2[w_n(x)]^2\,dx = 1,$$

and the sign has been fixed by the requirement that $w_n(1-x)^{-\frac{1}{2}}$ be positive for $x = 1$. (w_n vanishes like $(1-x)^{\frac{1}{2}}$ at $x = 1$.) For some purposes, it is more convenient to work with the related functions,

$$\Omega_n(x) = x(1-x^2)^{-\frac{1}{2}}\, w_n(x),$$

which satisfy the orthogonality relation,

$$\int_0^1 \Omega_n(x)\,\Omega_m(x)\,(1-x^2)\,dx = \delta_{nm}. \tag{47}$$

In terms of these functions, the angular velocity and stream function are

$$(w/r) \simeq w(1-x^2)^{-\frac{1}{2}} = \sum_{n=1}^{\infty} c_n(t) \exp(-\lambda_n \zeta)\,\Omega_n(x), \tag{48}$$

and
$$\psi = \frac{1-x^2}{2Bx} \sum_{n=1}^{\infty} \frac{1}{\lambda_n} \frac{dc_n}{dt} \exp(-\lambda_n \zeta)\,\Omega_n(x). \tag{49}$$

The coefficients $c_n(t)$ are determined by the boundary condition (41), by substituting the expansions (48) and (49). The result is

$$\frac{1}{B|x|^{\frac{1}{2}}} \sum_{n=1}^{\infty} \frac{1}{\lambda_n} \frac{dc_n}{dt}\,\Omega_n(x) + \sum_{n=1}^{\infty} c_n\,\Omega_n(x) = \Omega_B. \tag{50}$$

Because of the $|x|^{-\frac{1}{2}}$ factor, the modes are coupled. In principal one can obtain an infinite set of coupled equations for the coefficients c_n (first-order differential equations in time). These equations are then to be solved subject to given initial values $c_n(0)$, which are known whenever w is specified at $t = 0$. Fortunately this rather complicated calculation can be avoided in two cases of particular interest. Consider first impulsive spin-up. Then Ω_B is constant, and, although the transients are difficult to analyze, the final steady state is easily calculated. As $t \to \infty$, $(dc_n/dt) \to 0$ and $c_n \to c_n^{(\infty)}$, and we obtain from (47) and (50)

$$c_n^{(\infty)} = \Omega_B \int_0^1 (1-x^2)\,\Omega_n(x)\,dx. \tag{51}$$

These coefficients are easily calculated (appendix A) and the interior angular velocity is then calculated from the series (48). We can also obtain some information about the time required to reach steady state. The terms on the left of (50) are comparable in order-of-magnitude for $(dc_n/dt) \sim \lambda_n B c_n$, and this defines a time scale $t_n \sim c_n/(dc_n/dt) \sim (\lambda_n B)^{-1} \sim B^{-1}$. The dimensional time is

$$\hat{t} \sim \hat{\Omega}_0^{-1} E^{-\frac{1}{2}} B^{-1} \sim \hat{\Omega}_0^{-1} [\hat{\nu}/(\hat{\delta}^2 \hat{\Omega}_0)]^{-\frac{1}{2}}, \tag{52}$$

where $\hat{\delta} \sim (\hat{\Omega}_0/\hat{N})\hat{L}$ is the thickness of the spin-up layer. Thus, the spin-up time is given by the same formula as in the unstratified case, provided that the characteristic length is taken to be the spin-up layer thickness $\hat{\delta}$ (Walin 1969).

A second case which can be handled easily is that of slow continuous spin-up. If Ω_B changes on a time-scale much longer than that given by (52), then there is a quasi-steady state in which the Ekman process is able to keep up with the changes in the boundary velocity. Again we can neglect the time-derivatives in (50), and the coefficients c_n are given by (51), where Ω_B is now a slowly varying function of time. The small meridional circulations required can then be calculated from (49). One quantity of interest is the Ekman suction velocity, which is obtained from (12) and (49):

$$\mathbf{v}^{(I)}.\mathbf{n}|_{\zeta=0} = \frac{E^{\frac{1}{2}}(d\Omega_B/dt)}{2B}\frac{d}{dx}\left[-\frac{(1-x^2)}{x}\sum_{n=1}^{\infty}\frac{A_n\,\Omega_n(x)}{\lambda_n}\right], \tag{53}$$

where

$$A_n = \int_0^1 (1-x^2)\,\Omega_n(x)\,dx. \tag{54}$$

The actual numerical evaluation of the solution from these formulas is discussed in appendix A.

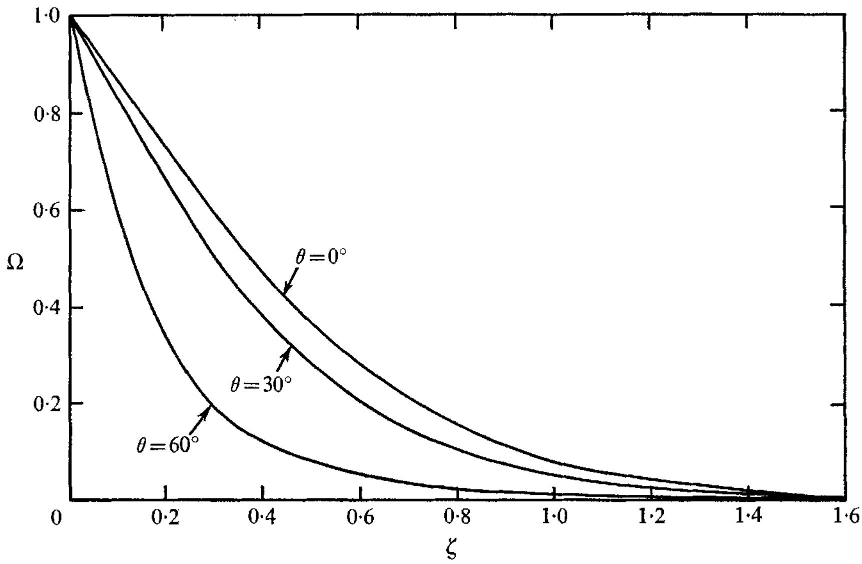

FIGURE 1. Interior angular velocity Ω as a function of depth ζ for colatitudes $\theta = 0°, 30°, 60°$.

3.2. *Discussion of solution*

The principal quantity of interest is the interior angular velocity, which may be written as

$$\Omega = (w/\Omega_B)\,(1-x^2)^{-\frac{1}{2}} = \sum_{n=1}^{\infty} A_n \exp\left(-\lambda_n\zeta\right)\Omega_n(x). \tag{55}$$

We have tabulated Ω as a function of ζ for the six angles $\theta = 0°\ (15°)\ 75°$. Figure 1 shows a plot of Ω *versus* ζ for $\theta = 0°$, $30°$ and $60°$. It is clear that the spin-up layer becomes thinner as the equator is approached. The characteristic thickness of the layer is

$$\hat{\delta} = \gamma\hat{R}(\hat{\Omega}_0|\cos\theta|/\hat{N}), \tag{56}$$

where γ is a dimensionless number of order one. If $\hat{\delta}$ is defined to be the distance at which Ω falls to 10 % of its value on the boundary, then our numerical results show that $\gamma = 1{\cdot}8$ for $0 \leqslant \theta \leqslant 60°$. Near the equator, γ becomes smaller; $\gamma = 1{\cdot}4$, for example, for $\theta = 75°$. The formula (56) is qualitatively like the results obtained by Holton (1965), Walin (1969), and Sakurai (1969*a*, *b*) for a circular cylinder, the major difference being the occurrence of the normal component of angular velocity ($\hat{\Omega}_0|\cos\theta|$) in the formula here.

For the case of continuous spin-up, the Ekman suction velocity is also of interest. We have calculated this quantity numerically on the basis of (53). We find that for spin-up, there is a flux into the Ekman layer for $0 \leqslant \theta \leqslant \theta_0$ and a return flow to the interior for $\theta_0 \leqslant \theta \leqslant 90°$, with $\theta_0 \simeq 66°$.

The solution obtained in §3.1 is based on the approximate equations (44) and (45). A more systematic approach is to introduce a formal expansion in powers of B^{-1} into the full equations (42) and (43). Equations (44) and (45) then appear as the lowest order equations in the sequence generated by the expansion. Although the solution given in §3.1 is adequate for most purposes, the higher-order terms are not without interest. Consider, for example, the angular velocity in the equatorial plane for the case of impulsive spin-up. The theory of §3.1 predicts that it is zero in the interior ($\zeta > 0$) and Ω_B on the boundary ($\zeta = 0$). (A similar discontinuity occurs in the solutions for a cylinder given by Walin (1969) and Sakurai (1969*a*, *b*).) Detailed calculations based on the expansion in powers of B^{-1} show that

$$\Omega(\zeta, \theta = \tfrac{1}{2}\pi, t = \infty) = B^{-1} \sum_{n=1}^{\infty} c_n^{(\infty)} \lambda_n^{-1} w_n'(0) \exp(-\lambda_n \zeta) + O(B^{-2}). \qquad (57)$$

Thus the interior angular velocity in the equatorial plane is small (of the order of B^{-1}) but not zero. Since $\Omega = \Omega_B$ on the boundary at all latitudes, it follows that the gradient of the angular velocity becomes large near the equator.

4. Remarks on the solar spin-down problem

Dicke (1964) and Roxburgh (1964) have suggested that the interior of the sun may be rotating more rapidly than the surface layers, a hypothesis considerably strengthened by the oblateness measurements of Dicke & Goldenberg (1967). Howard, Moore & Spiegel (1967) suggested that such a state of differential rotation would not persist because of the Ekman spin-down process. The subsequent controversy has produced many qualitative arguments, but no calculation of the Ekman process under solar conditions. The essential features of the solar problem are spherical geometry, strong stratification, and very small Prandtl number. The present work fails with respect to the third condition, since we have taken the Prandtl number to be of order one with respect to the Ekman number. Nevertheless, it is still of interest to consider the numbers for the solar case. The 'container wall' in the case of the sun is the interior interface between the radiative core and the bottom of the convection zone, so that $\hat{L} \sim 5 \times 10^5$ km, the radius of the radiative core. For the interior angular velocity, we take the value suggested by Dicke (1967*a*), $\hat{\Omega}_0 \sim 3 \times 10^{-5}\,\mathrm{sec}^{-1}$. The Brunt–Väisälä frequency may be calculated from a solar model, and a typical value in the vicinity of the convection zone boundary is $\hat{N} \sim 10^{-3}\,\mathrm{sec}^{-1}$. For these numbers, the 10 % scale (56) is (for $0 \leqslant \theta \leqslant 60°$)

$$\hat{\delta} \sim 2{\cdot}7 \times 10^4 \cos\theta\,\mathrm{km}.$$

At the pole, for example, $\hat{\delta} \sim 27{,}000$ km, while at $\theta = 45°$, $\hat{\delta} \sim 19{,}000$ km. These results suggest that for slow, continuous spin-down of the solar convection zone, the modification of the internal rotation state is confined to a layer below the convection zone $\sim$ 30,000 km thick. However, the very large (radiative) thermal diffusivity of the solar material may well alter this conclusion. To get reliable information about the Ekman process under solar conditions, it will be necessary to develop a spin-down theory for the case of very small Prandtl number $\mathscr{P}$ (for example, $\mathscr{P} = O(E^{\frac{1}{2}})$).

Finally, we note that the present calculations predict a large gradient of azimuthal velocity near the equator. If rotational instabilities of the type discussed recently by Goldreich & Schubert (1967) are operative, then, in a continuous spin-down process, the instabilities should appear first at the equator. (The occurrence of the rotational instability in the sun is still a point of controversy. Objections have been raised by Clark *et al.* (1969) and Dicke (1967*b*, 1970).)

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Appendix A. Computation of the eigenfunctions and interior flow

Computation of the eigenfunctions

Equation (46) has a singular point at $x = 1$. The indicial equation has roots $\pm\frac{1}{2}$, so that the acceptable solution behaves like $(1-x)^{\frac{1}{2}}$; thus, we require that $(1-x^2)(dw_n/dx)$ be bounded at $x = 1$. The condition at $x = 0$ is $w_n(0) = 0$. By use of a standard theorem on singular Sturm–Liouville systems (see for example, Weinberger 1965, p. 176), it is not hard to show that the spectrum is discrete and the eigenfunctions are complete. Although equation (46) is the equation for oblate spheroidal angle functions, we have not found the results available in the literature useful here, because our eigenvalue problem is somewhat different from the 'standard' eigenvalue problem for spheroidal wave functions. A direct numerical approach is difficult because of the singularity at $x = 1$. The procedure used here—an expansion in Legendre functions—is suggested by the form of the equation. The functions $p_n^m(x)$, for $m = 1$, are eigenfunctions of the operator on the left-hand side of (46). Since w_n is an odd function of x, we need p_n^1 only for n even. Thus, we try

$$w_n(x) = \sum_{j=1}^{\infty} D_j^{(n)} P_{2j}^1(x), \tag{A 1}$$

where

$$P_{2j}^1(x) = \frac{(1-x^2)^{\frac{1}{2}}}{(2j)!\,2^{2j}} \frac{d^{2j+1}}{dx^{2j+1}} (x^2-1)^{2j}. \tag{A 2}$$

Substitution into (46) yields

$$\sum_{j=1}^{\infty} 2j(2j+1) D_j^{(n)} P_{2j}^1 = \lambda_n^2 x^2 \sum_{j=1}^{\infty} D_j^{(n)} P_{2j}^1, \tag{A 3}$$

and it is clear that the coefficients are coupled by the x^2 term. An infinite matrix equation is obtained by multiplying by P_{2k}^1 and integrating over [0, 1]. The result is conveniently written as

$$\sum_{j=1}^{\infty} M_{kj} C_j = \beta C_k, \tag{A 4}$$

where $\beta = \lambda_n^{-2}$ and $\qquad C_j = 2j(2j+1)(4j+1)^{-\frac{1}{2}} D_j^{(n)},$

with M being the symmetric matrix

$$M_{kj} = \frac{[(4k+1)(4j+1)]^{\frac{1}{2}}}{(2k)(2k+1)(2j)(2j+1)} \int_0^1 x^2 P_{2k}^1 P_{2j}^1 \, dx.$$

The matrix elements M_{kj} may be evaluated with the help of recurrence formulas for Legendre functions (Abramowitz & Stegun 1964). One finds the selection rule $M_{kj} = 0$ unless $j = k-1$, k or $k+1$, and the values

$$M_{kk} = \frac{8k^2+4k-3}{(2k)(2k+1)(4k-1)(4k+3)}$$

and $\qquad M_{k,k+1} = M_{k+1,k} = [(4k+1)(4k+3)^2(4k+5)]^{-\frac{1}{2}}.$

The infinite set of homogeneous equations (A 4) determines eigenvectors $C_j^{(n)}$ and eigenvalues β_n. Truncation of the set of equations leads to an algebraic eigenvalue problem, which can be solved numerically. We have obtained the eigenvalues by the method of bisection, and the eigenvectors by inverse iteration, as described in detail by Wilkinson (1965). All calculations were carried out double precision, and various accuracy checks indicated a high degree of precision for the truncated eigenvalue problem. The effect of truncation was assessed by repeating the calculations for different matrix sizes (from 20×20 to 100×100). The final calculations reported here were based on 30 eigenfunctions calculated from an 80×80 matrix. The amplitudes of the eigenvectors are fixed by the normalization condition, which may be written as

$$\int_0^1 x^2 [w_n(x)]^2 \, dx = \sum_{k,j} M_{kj} C_k^{(n)} C_j^{(n)} = \beta_n \sum_k [C_k^{(n)}]^2 = 1.$$

In the computation of the interior angular velocity, we have found it convenient to use the functions,

$$\Omega_n(x) = x(1-x^2)^{-\frac{1}{2}} w_n(x),$$

which can be expressed simply in terms of the ordinary Legendre polynomials:

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2-1)^n. \tag{A 5}$$

From equations (A 1), (A 2), (A 5), and the recurrence formula (Abramowitz & Stegun 1964)

$$(1-x^2)(dP_{2k}/dx) = 2k(P_{2k-1} - xP_{2k}),$$

one can show that

$$\Omega_n(x) = \frac{2x}{1-x^2} \sum_{k=1}^{\infty} D_k^{(n)} k [P_{2k-1}(x) - xP_{2k}(x)]. \tag{A 6}$$

The expression (A 6) is indeterminate for $x = 1$, and it can be shown that

$$\Omega_n(1) = \sum_{k=1}^{\infty} D_k^{(n)} k(2k+1). \tag{A 7}$$

Equations (A 6) and (A 7) reduce the problem of computing the eigenfunctions to the problem of computing Legendre polynomials. We have used these formulas to compute $\Omega_n(x)$ for $n = 1, 2, \ldots, 30$. It is perhaps worth noting the following basic properties of the eigenfunctions: (i) they vanish at $x = 0$; (ii) they oscillate least rapidly near $x = 0$ and most rapidly near $x = 1$; (iii) $\Omega_n(1) \sim (\pi^2/2)(n+\frac{1}{8})^{\frac{3}{2}}$ as $n \to \infty$; (iv) away from $x = 1$, all the eigenfunctions are of order one.

Computation of the interior flow

The 30 eigenfunctions available have been used to compute the interior angular velocity as a function of ζ (distance from the boundary) and θ (colatitude). The series to be summed is given by (55). The coefficients A_n (equation (54)) are given by

$$A_n = \int_0^1 (1-x^2)\,\Omega_n(x)\,dx = (2D_1^{(n)}/5).$$

These coefficients diminish rather slowly with n; for example,

$$A_{30} = -0{\cdot}04044629,$$

so that the convergence of the series (55) for Ω is a matter for careful consideration. Numerical exploration has shown that 30 terms are not sufficient for an accurate result when $\zeta = 0$; the exponential factors are needed for convergence. For $\theta = 0°$, for example, 4-place accuracy is attained for $\zeta \geqslant 0{\cdot}15$ and 3-place accuracy for $\zeta \geqslant 0{\cdot}12$. Fortunately, the calculated values for $0{\cdot}15 \leqslant \zeta \leqslant 0{\cdot}20$ are on a straight line; furthermore, this straight line extrapolated passes through the point $1{\cdot}0000$ for $\zeta = 0{\cdot}0$. Thus, linear extrapolation can be used to complete the calculation of Ω. We estimate that the values calculated in this way are accurate to 3 places for $\theta \leqslant 60°$. Although tables of Ω for $\theta = 0°\,(15°)\,75°$ have been constructed, they are not given here since the main features of interest are already present in figure 1.

Asymptotic calculation of eigenfunctions

When λ is large, approximate solutions of (46) can be obtained by the WKB method. This requires a matching of an interior solution to a solution valid near the singular point ($x = 1$) and another solution valid near the turning point ($x = 0$).

Consider first the interior solution. The standard approximation (Morse & Feshbach 1953, p. 1092) gives

$$w = w_I = c_I[x^2(1-x^2)]^{-\frac{1}{4}}\cos[\lambda\{1-(1-x^2)^{\frac{1}{2}}\}-\phi], \tag{A 8}$$

where c_I and ϕ are constants. As $x \to 0$,

$$w_I \underset{x\to 0}{\simeq} c_I x^{-\frac{1}{2}}\cos[(\lambda x^2/2)-\phi], \tag{A 9}$$

and we see that the assumptions behind the WKB method break down for $x = O(\lambda^{-\frac{1}{2}})$. This suggests the scaling $y = \lambda^{\frac{1}{2}}x$, which gives the following equation for w:

$$(d^2w/dy^2) + y^2 w = O(\lambda^{-1}).$$

The solution obtained by ignoring the right-hand side is

$$w_0 = c_0 y^{\frac{1}{2}} J_{\frac{1}{4}}(y^2/2),$$

where $J_{\frac{1}{4}}$ is a Bessel function and c_0 is a constant. The asymptotic behaviour of w_0 for large y is (Abramowitz & Stegun 1964)

$$w_0 \underset{y\to\infty}{\simeq} 2c_0(\pi y)^{-\frac{1}{2}} \cos\left[(y^2/2)-(3\pi/8)\right]. \tag{A 10}$$

The matching of w_I and w_0 is accomplished by comparing (A 9) and (A 10) with the result

$$c_0 = \tfrac{1}{2}(-1)^k \pi^{\frac{1}{2}}\lambda^{\frac{1}{4}} c_I, \quad \phi = (3\pi/8)+k\pi \tag{A 11}$$

where k is an integer.

Near $x = 1$, the solution also breaks down. We have

$$w_I \underset{x\to 1}{\simeq} c_I[2(1-x)]^{-\frac{1}{4}} \cos\left[\lambda\{1-(2-2x)^{\frac{1}{2}}\}-\phi\right]. \tag{A 12}$$

This suggests the scaling $s = \lambda^2(1-x)$, which yields for w the equation

$$\frac{d^2w}{ds^2}+\frac{1}{s}\frac{dw}{ds}+\frac{2s-1}{4s^2}\,w = O(\lambda^{-2}).$$

The solution obtained by ignoring the right-hand side is

$$w = w_1 = c_1 J_1([2s]^{\frac{1}{2}}),$$

where c_1 is a constant and J_1 is a Bessel function. The asymptotic behaviour of w_1 for large s is

$$w_1 \underset{s\to\infty}{\simeq} c_1(\pi^2 s/2)^{-\frac{1}{4}} \cos\left[(2s)^{\frac{1}{2}}-(3\pi/4)\right]. \tag{A 13}$$

The matching of w_I and w_1 (equations (A 12) and (A 13)) yields

$$c_1 = (-1)^m (\pi\lambda/2)^{\frac{1}{2}} c_I, \quad \phi = m\pi-(3\pi/4)+\lambda, \tag{A 14}$$

where m is an integer. Elimination of ϕ between (A 11) and (A 14) gives an asymptotic formula for the eigenvalues:

$$\lambda_n = (n+\tfrac{1}{8})\,\pi, \tag{A 15}$$

where n is a positive integer. This formula is surprisingly accurate. Even for $n = 1$, the difference is less than $\frac{1}{2}$ %, the numerical value being $\lambda_1^2 = 12{\cdot}5430$ and the asymptotic value being $(9\pi/8)^2 = 12{\cdot}4912$. The percentage error is even less for higher eigenvalues. For $n = 10$, for example, the numerical value is $\lambda_{10}^2 = 1012{\cdot}2446$ and the asymptotic value is $\lambda_{10}^2 = (81\pi/8)^2 = 1011{\cdot}7887$.

As a further check on the numerical results, we have also compared asymptotic and numerical values of the eigenfunctions. The asymptotic formulas for the eigenfunctions can be obtained from the formulas above and the fact that $c_I = \sqrt{2}$, which is established from the normalization condition,

$$\int_0^1 x^2[w_n(x)]^2\,dx = 1,$$

and the approximation (A 8). The results are

$$\Omega_n(x) \simeq \left[\frac{4x^2}{(1-x^2)^3}\right]^{\frac{1}{4}} \cos\left[\lambda_n(1-x^2)^{\frac{1}{2}}-(3\pi/4)\right], \quad (0 < x < 1), \tag{A 16}$$

$$\Omega_n(x) \simeq (-1)^{n+1}(\pi\lambda_n x^2)^{\frac{1}{2}} J_{\frac{1}{4}}(\lambda_n x^2/2), \quad (x = O(\lambda_n^{-\frac{1}{2}})), \tag{A 17}$$

and $$\Omega_n(x) \simeq \left[\frac{\pi\lambda_n x^2}{2(1-x)}\right]^{\frac{1}{2}} J_1(\lambda_n[2(1-x)]^{\frac{1}{2}}), \quad (1-x = O(\lambda_n^{-2})). \tag{A 18}$$

In particular, the values of the eigenfunctions at the pole ($x = 1$) may be calculated from equation (A 18):

$$\Omega_n(1) \simeq \tfrac{1}{2}(\pi\lambda_n^3)^{\frac{1}{2}} \simeq \tfrac{1}{2}\pi^2(n+\tfrac{1}{8})^{\frac{3}{2}}. \tag{A 19}$$

The agreement between the asymptotic values given by (A 19) and the numerical values given by (A 7) is exceptionally good. Even for $n = 1$, the discrepancy is less than 1 % (numerical = 5·9207, asymptotic = 5·8884). For higher n, the agreement is even better (for $n = 10$, numerical = 159·0304, asymptotic = 158·9873).

Although we have prepared numerical tables of the functions occurring in the present work, they are not included here, since the essential features of the solution are already contained in figure 1. Readers interested in more details are invited to write to the authors.

Appendix B. The neglect of compressibility

The continuity equation (in dimensional form) may be written as

$$(\partial\hat{\rho}/\partial\hat{t}) + \hat{\mathbf{v}}\,.\,\hat{\nabla}\hat{\rho}_0 + \hat{\rho}_0\hat{\nabla}\,.\,\hat{\mathbf{v}} = 0, \tag{B 1}$$

and we wish to show that the first two terms are much smaller than the third. We do this by a sequence of estimates in which we express the order of magnitude of all relevant quantities in terms of any convenient one, which we choose to be the azimuthal velocity $\hat{w}$. The basic time scale is $\hat{t} \sim \hat{\Omega}_0^{-1}E^{-\frac{1}{2}}(\hat{\delta}/\hat{L})$, and the basic length scales are $\hat{L}$ (along equipotentials) and $\hat{\delta}$ (normal to equipotentials). In order of magnitude, the scale height is $\hat{H} \sim \hat{c}_0^2/\hat{g}$, and we also have

$$\hat{N}^2/\hat{g} \sim |\hat{\nabla}\hat{s}_0|/\hat{c}_{p_0} \sim \hat{H}^{-1}.$$

We start with the dimensional form of equation (24):

$$2\hat{\rho}_0\hat{w}\hat{\mathbf{\Omega}}_0 \times \mathbf{e}_\phi = \hat{\nabla}\hat{p} + \hat{\rho}\hat{\nabla}\hat{\Phi}_0. \tag{B 2}$$

From the component of (B 2) along an equipotential, we get

$$\hat{p} \sim \hat{\rho}_0\hat{\Omega}_0\hat{L}\hat{w}.$$

From this equation and the component normal to equipotentials, one gets

$$\hat{\rho} \sim \hat{\rho}_0(\hat{\Omega}_0\hat{L}/\hat{\delta}\hat{g})\hat{w}.$$

From the thermodynamic relation (7) and the above equations, it follows that

$$\hat{s} \sim (\hat{c}_{p_0}/\hat{\beta}_0\hat{T}_0)\,(\hat{\rho}/\hat{\rho}_0) \sim (\hat{c}_{p_0}/\hat{\beta}_0\hat{T}_0)\,(\hat{\Omega}_0\hat{L}/\hat{\delta}\hat{g})\hat{w}, \tag{B 3}$$

since the term $\hat{p}$ in (7) is smaller than $\hat{c}_0^2\hat{\rho}$ by the factor $\hat{\delta}/\hat{H}$. The dimensional form of the entropy equation (26) is

$$(\partial\hat{s}/\partial\hat{t}) + \hat{\mathbf{v}}\,.\,\hat{\nabla}\hat{s}_0 = 0. \tag{B 4}$$

The velocity involved in (B 4) is the velocity normal to an equipotential, $\hat{v}_\perp$; and it follows from (17), (B 3), and (B 4), that

$$\hat{v}_\perp \sim E^{\frac{1}{2}}\hat{w}.$$

The component of velocity along an equipotential is obtained from the azimuthal equation

$$(\partial\hat{w}/\partial\hat{t}) = -2\hat{\Omega}_0\mathbf{e}_r\,.\,\hat{\mathbf{v}},$$

from which one can show that $\hat{v}_{\parallel} \sim (\hat{L}/\hat{\delta})E^{\frac{1}{2}}\hat{w}$.

The estimates of the three terms in the continuity equation can be put into the following form:

$$(\partial\hat{\rho}/\partial\hat{t}) \sim E^{\frac{1}{2}}\hat{\rho}_0\hat{w}/\hat{H}, \quad \hat{\mathbf{v}}.\hat{\nabla}\hat{\rho}_0 \sim E^{\frac{1}{2}}\hat{\rho}_0\hat{w}/\hat{H}$$

and

$$\hat{\rho}_0\hat{\nabla}.\hat{\mathbf{v}} \sim E^{\frac{1}{2}}\hat{\rho}_0\hat{w}/\hat{\delta}.$$

Thus, the error caused by neglecting compressibility is of the order of $(\hat{\delta}/\hat{H})$.

The approximation also must be justified within the boundary layers; it turns out that the neglected terms are actually higher-order in the Ekman number. The estimates are straightforward, so we do not give them here.

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